# Tracking Control of a Nonminimum Phase Inverted Pendulum

**Linqi Ye, Qun Zong, Xiuyun Zhang, Dandan Wang and Qi Dong**

**Abstract** Three methods are investigated for the tracking problem of the famous cart–pole system (a kind of planar inverted pendulum). The output is required to track a sinusoid signal. Control design is based on the linearized model. First, we show that using output error and states feedback, approximate tracking can be achieved with bounded tracking error. Then exact tracking via output regulation is investigated. By constructing a regulator equation, the equivalent input and equivalent states which are needed to maintain output at the reference trajectory can be calculated. We show that the tracking problem is equivalent to the stabilizing problem in the states error coordinate. Finally, we study exact tracking via stable system center method. Because of the nonminimum phase property, a bounded solution for the internal dynamics is required and is estimated by stable system center method. Then the tracking problem can also be transformed into a stabilizing problem. Simulations are made for each method.

**Keywords** Tracking control · Nonminimum phase · Output regulation · Stable system center · Inverted pendulum

## 1 Introduction

Tracking control is much more complicated than stabilizing control, especially when the system is nonminimum phase. Stabilizing control only requires maintaining the output at a set-point, i.e., the equilibrium. For linear system, the stabilizing control theory has already been very perfect. Through proper coordinate change, the equilibrium can be moved to zero, and then various linear control

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methods such as pole assignment and LQR can be used. So to say, it is not difficult for an engineer with basic control knowledge to accomplish the task of stabilizing a linear system. However, when it turns to the tracking problem, which aims at keeping the output moving along a varying trajectory (such as a sinusoid signal), things become much more challenging even for linear system.

The most widely applied methods, PID control and the basic feedback control method, and pole assignment, are not tracking control methods. They can be used to set-point control and achieve zero steady-state error, but when used in tracking control, the error increases as the tracking signal frequency goes up. To eliminate the tracking error, researchers have done amount of work in the past years. Summarizing the existed literatures, we may find that inversion seems necessary for exact tracking. Recall stabilizing control, the equilibrium to be achieved is known, which means we know the value every state should reach. Finally, the output will reach the set-point and other states also reach their steady values. What about tracking control? When the output exactly moves along the reference trajectory, we can imagine that other states should also move along certain trajectories. So this is the case for exact tracking: every state does not converge to fixed point but converges to fixed trajectory. In other words, all states as well as the input have specific trajectories to maintain the output at the reference trajectory. If we can calculate these trajectories, then tracking problems can be converted into stabilizing problem. This idea is the key for exact tracking.

In a manner of speaking, exact tracking is indeed an inversion problem, that is, giving a plant and a reference output, to calculate the equivalent input and equivalent states. A similar concept, named as stable inversion, has been widely studied [1, 2]. However, what we focused on is another method, output regulation theory [3, 4], which aims to achieve asymptotic tracking and disturbance rejection for a class of reference trajectories and disturbances. Suppose the reference output is generated by a known exosystem; then regulator equation can be constructed based on the plant and exosystem. The solution of the regulator equation directly leads to equivalent input and equivalent states. It should be noted that, whether the system is minimum phase or nonminimum phase, it has no influence on using output regulation theory.

The application of differential geometry theory to nonlinear control [5] brings us a new perspective to look at the problem of tracking control. Now we know that a system is composed of external and internal states. The dynamics of the internal states, i.e., the internal dynamics, is only driven by the external states. When it comes to minimum phase system, the internal dynamics is naturally stable, thus we only need to design tracking controller for the external states. This is not difficult because the external states are derivatives of the output which implies the equivalent external states are simply the derivatives of the reference output. But for nonminimum phase system, the internal dynamics is not stable, which means the internal states will not converge to their equivalent trajectories naturally. We need to force it. The equivalent internal states value is a bounded solution of the internal dynamics, and is also called ideal internal dynamics (IID). Stable system center [6, 7] is a method to estimate the IID asymptotically. When IID is obtained, we then know the equivalent values of all states and the tracking problem can be transformed into stabilizing problem.

Inverted pendulum is a classical nonlinear, unstable, nonminimum phase, underactuated system which has been widely used in control education and research to demonstrate the effectiveness of various control methods. References [8–13] have studied the stabilizing problem of inverted pendulum while the tracking problem has been studied in [14–16]. Here we will use it as an example to study some tracking control methods. Our methods are based on the linearized model.

## 2 Model Description

The cart–pole system consists of a cart and an inverted pendulum as shown in Fig. 1.

Here $u$ is an external force that moves the cart in the horizontal plane, $s$ is the cart position, $\theta$ is the pole angle, $M$ is the mass of cart, $m$ is the mass of pole, and $l$ is the half length of pole. According to [8], the motion equations of the inverted pendulum are

$$\begin{aligned} &(M+m)\ddot{s} - ml\cos\theta\ddot{\theta} + ml\sin\theta\ddot{\theta}^2 = u \\ &I\ddot{\theta} - ml\cos\theta\ddot{s} - mgl\sin\theta = 0 \end{aligned} \tag{1}$$

where $I = 4ml^2/3$ is the moment of inertia of the pendulum with respect to the pivot. Define $v = \dot{s}, \omega = \theta$ as the cart velocity and angle acceleration, where (1) can be written in state space form

$$\begin{aligned} &\dot{s} = v, \quad \dot{v} = \frac{mg\sin\theta\cos\theta - 4ml\omega^2\sin\theta/3}{4(M+m)/3 - m\cos^2\theta} + \frac{4/3}{4(M+m)/3 - m\cos^2\theta}u \\ &\ddot{\theta} = \omega, \quad \dot{\omega} = \frac{(M+m)g\sin\theta - ml\omega^2\sin\theta\cos\theta}{4(M+m)l/3 - ml\cos^2\theta} + \frac{\cos\theta}{4(M+m)l/3 - ml\cos^2\theta}u \end{aligned} \tag{2}$$

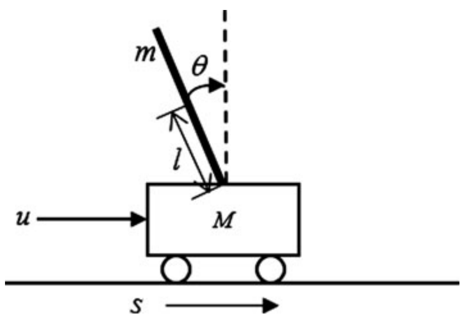



**Fig. 1** Cart–pole system

where $g = 9.8\ \text{m/s}^2$ is the acceleration due to gravity, and other parameters are selected as $m = 1\,\text{kg}, M = 10\,\text{kg}, l = 1\,\text{m}$.

It can be verified that $(s^*, 0, 0, 0)$ is the equilibrium set (i.e., the system is in equilibrium when all states derivatives equal zero). The linearized model of the nonlinear model (2) in the equilibrium $(s^*, 0, 0, 0)$ is

$$\begin{aligned} \dot{s} &= v, \quad \dot{v} = 0.7171\theta + 0.09756u \\ \theta &= \omega, \quad \dot{\omega} = 7.888\theta + 0.07317u \end{aligned} \tag{3}$$

This linearized model will be used later for tracking controller design, while the nonlinear model is used for simulation. The control objective is to track sinusoid trajectory for the cart position and pole angle, respectively. Two cases are considered here:

**Case 1**: Tracking control of the cart position with the desired trajectory $s_d = \sin(t)$.
**Case 2**: Tracking control of the pole angle with the desired trajectory $\theta_d = 0.2\sin(t)$.

## 3 Approximate Tracking Control

Consider the following nth-order SISO linear system:

$$\dot{x} = Ax + Bu, \quad y = x_1 \tag{4}$$

where $x = [x_1, x_2, \ldots, x_n]^T$ is the states vector, $u, y \in R^1$ are the input and the output, respectively. Suppose the desired tracking trajectory is $y_d$, where $y_d$ and $\dot{y}_d$ are bounded, and denote the tracking error as $\tilde{x}_1 = y - y_d$.

**Theorem 1** *If the feedback control $u = -Kx$ can stabilize system (4), then the control law*

$$u = -K\tilde{x} \tag{5}$$

*with $\tilde{x} = [\tilde{x}_1, x_2, \ldots, x_n]^T$ can achieve approximate tracking with bounded tracking error for system (4).*

*Proof* First, it indicates that $A - BK$ is Hurwitz since $u = -Kx$ can stabilize system (4). And if control law $u = -K\tilde{x}$ is applied, the closed-loop system becomes

$$\dot{\tilde{x}} = (A - BK)\tilde{x} + d \tag{6}$$

where $d = AY_d - \dot{Y}_d$ with $Y_d = [y_d, 0, \ldots, 0]^T$ can be seen as a perturbation. Since $A - BK$ is Hurwitz, according to the input-to-state stability theorem [17], $\tilde{x}$ will

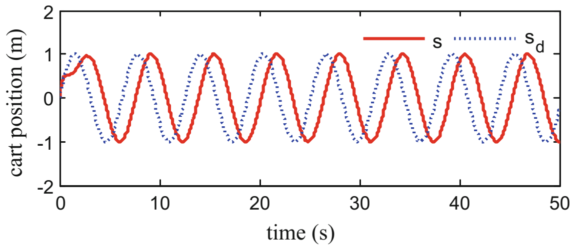



**Fig. 2** Approximate tracking of cart position

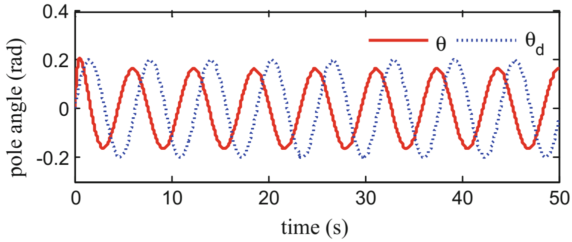



**Fig. 3** Approximate tracking of pole angle

remain bounded for bounded $d$. Thus approximate tracking is achieved. This completes the proof.

*Remark* For the inverted pendulum model (4), the first column of $A$ equals zero which indicates that $d = -\dot{Y}_d$. Thus when $y_d$ is a constant, i.e., $d = 0$, control law $u = -K\tilde{x}$ can achieve asymptotic tracking. However, as the frequency of $y_d$ increases, which means the upper bound of $d$ increases, the tracking error will go up.

For case 1, we have $\tilde{x} = [s - s_d, v, \theta, \omega]^T$, and the control gain is selected as $K = [-200.8, \quad -248.2, \quad 1851.6, \quad 590.6]$ to place the closed-loop poles at $[-8, -9, -1 \pm i]$. The simulation result is shown in Fig. 2, from which we can see there exists a serious phase lag.

For case 2, we have $\tilde{x} = [s, v, \theta - \theta_d, \omega]^T$ and $K$ is the same as that in case 1. The simulation result is presented in Fig. 3. Also a phase lag is observed.

## 4 Exact Tracking Control via Output Regulation

As mentioned in the former section, using output error and states feedback, approximate tracking with bounded tracking error can be achieved. In this section we will illustrate that using output regulation theory, output tracking problem can be transformed into a stabilizing problem.

First, let us consider how to transform a tracking problem into a stabilizing problem. Consider the nth-order SISO linear system as follows:

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{7}$$

where $x = [x_1, x_2, \ldots, x_n]^T$ is the states vector, $u, y \in R^1$ are the input and output, respectively.

Suppose the control objective is to let $y$ track $y_d$. To maintain $y$ at $y_d$, we need certain quantity of control input and the states should also maintain in certain trajectories. Figuratively, we may call them equivalent input and equivalent states, and denote by $x_d, u_d$, respectively. Obviously, $x_d, u_d$ should satisfy the following equation:

$$\dot{x}_d = Ax_d + Bu_d, \quad y_d = Cx_d \tag{8}$$

Define the states tracking error $\tilde{x} = x - x_d$, output tracking error $e = y - y_d$, and new control input $v = u - u_d$. Subtracting (8) from (7), we can obtain the stabilizing form of the original tracking problem:

$$\dot{\tilde{x}} = A\tilde{x} + Bv, \quad e = C\tilde{x} \tag{9}$$

Therefore, the original tracking problem to let $y$ track $y_d$ is equivalent to the stabilizing problem of system (9). And it is interesting to note that system (9) has identical form to the original system (7). So we may say that the tracking problem of a linear system is nothing different from its stabilizing problem; the only thing we need to do is to calculate the equivalent input and equivalent states, and output regulation theory can help us realize it.

To use output regulation theory, we need to know the exosystem, which generates the reference output. We may suppose the exosystem is

$$\dot{w} = Sw, \quad y_d = Qw \tag{10}$$

Then the tracking problem is equivalent to the following output regulation problem:

$$\dot{x} = Ax + Bu, \quad e = Cx - Qw \tag{11}$$

The equivalent input and equivalent states $u_d, x_d$ should meet

$$\dot{x}_d = Ax_d + Bu_d, \quad e = Cx_d - Qw \tag{12}$$

Assume the equivalent input and equivalent states are $x_d = Xw, u_d = Uw$, where $X, U$ are matrices needed to be solved. Substituting $x_d = Xw, u_d = Uw$ into (12) yields the regulator equation:

$$XS = AX + BU, \quad 0 = CX - Q \tag{13}$$

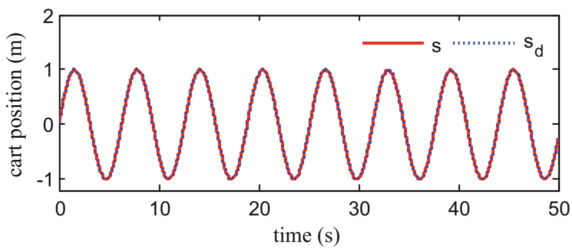



**Fig. 4** Cart position via output regulation

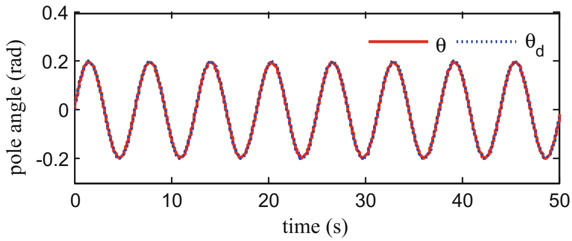



**Fig. 5** Pole angle via output regulation

It is a matrix equation and can be easily solved, and then we can obtain the equivalent input and equivalent states.

As for the stabilizing of system (9), we still use pole assignment. Letting $v = -K\tilde{x}$ with $\tilde{x} = x - x_d$, and $K$ ensures $A - BK$ be Hurwitz. Combine $v = u - u_d$ we can write the control law as

$$u = -K\tilde{x} + u_d \tag{14}$$

This yields the asymptotic stable closed-loop system $\dot{\tilde{x}} = (A - BK)\tilde{x}, \quad e = C\tilde{x}$ . Comparing (14) with (5), we can find that the asymptotic control law adds an equivalent input and use error feedback of all states.

Now let us turn to the tracking problem of the inverted pendulum. The exosystems in case 1 and case 2 are the same: $\dot{w}_1 = w_2, w_2 = -w_1$.

For case 1, we have $s_d = w_1$, i.e., $Q = [1 \quad 0]$. The solution of the regulator Eq. (13) is $X = [1, 0;\ 0, 1;\ 0.0898,\ 0; 0,\ 0.0898], U = [-10.9102,\ 0]$. Then control law (14) is applied with the same $K$ as before. The simulation result is shown in Fig. 4. We can see that almost perfect tracking is achieved.

For case 2, we have $\theta_d = 0.2w_1$, i.e., $Q = [0.2 \quad 0]$. The solution of the regulator Eq. (13) is $X = [2.2267\ ,\ 0\ ;\ \ 0,\ 2.2267\ ;\ 0.2,\ 0\ ;\ 0,\ 0.2], U = [-24.2933,\ 0]$. And also the control law (14) is applied with the same $K$ as before. The simulation result is shown in Fig. 5. The tracking result is also very perfect.

## 5 Exact Tracking Control via Stable System Center

Unlike output regulation method, which directly based on system (7) and solve all the equivalent states and equivalent input by a regulator equation, we will consider how to solve the tracking problem based on the normal form in this section. First, let us introduce how to turn system (7) into a normal form.

For system (7), if $CA^{i-1}B = 0(i = 1, \ldots, r-1), CA^{r-1}B \neq 0$. Then we say system (7) has relative degree r, and $y, \dot{y}, \ldots, y^{(r-1)}$ are called external states. By taking the following coordinate change,

$$\xi_i = y^{(i-1)} = CA^{i-1}x, i = 1, 2, ..., r \ ; \quad \eta_i \begin{cases} \in \{x_1, x_2, ..., x_n\} \\ \notin span\,\{\xi_1, \xi_2, ..., \xi_r\} \end{cases}, \quad i = 1, 2, ..., n-r \tag{15}$$

System (7) can be converted into the normal form as

$$\begin{aligned} y^{(r)} &= E\xi + F\eta + \alpha u \\ \dot{\eta} &= R\xi + S\eta + Tu \end{aligned} \tag{16}$$

where $\xi = [\xi_1, \xi_2, \ldots, \xi_r]^T$ is the external states vector and $\eta = [\eta_1, \eta_2, \ldots, \eta_{n-r}]^T$ is the internal states vector.

For simplicity, denote $z = [\xi, \eta]^T$; then the normal form (16) can be written as

$$\dot{z} = A_1 z + B_1 u, \quad y = z_1 \tag{17}$$

According to the first equation of (16), when $y$ moves along $y_d$, the equivalent input is given by

$$u_d = \alpha^{-1}\left(y_d^{(r)} - E\xi_d - F\eta_d\right) \tag{18}$$

with $\xi_d = \left[y_d, \dot{y}_d, \ldots, y_d^{(r-1)}\right]$. Substituting (18) into the second equation of (16) yields

$$\dot{\eta}_d = R\xi_d + S\eta_d + \alpha^{-1}T\left(y_d^{(r)} - E\xi_d - F\eta_d\right) \tag{19}$$

Denoting

$$Q = S - \alpha^{-1}TF, \quad r(t) = R\xi_d + \alpha^{-1}T\left(y_d^{(r)} - E\xi_d\right) \tag{20}$$

then (19) becomes

$$\dot{\eta}_d = Q\eta_d + r(t) \tag{21}$$

This is known as the internal dynamics. It seems that by giving any initial value $\eta_d(0)$ we can obtain a solution $\eta_d(t)$ through (21). But it should be noted that for nonminimum phase system, system (21) is unstable and cannot be used to generate a bounded $\eta_d(t)$. So this is the case: we need to calculate a bounded solution $\eta_d(t)$ that satisfies (21). This bounded solution is also called ideal internal dynamics (IID). Stable system center [6, 7] provides a method to estimate the IID.

Suppose the eigenpolynomial of the exosystem which generates the reference output $y_d$ is

$$s^k + p_{k-1}s^{k-1} + \ldots + p_1 s + p_0 \tag{22}$$

Then the IID can be estimated by constructing the following causal equation

$$\hat{\eta}_d^{(k)} + c_{k-1}\hat{\eta}_d^{(k-1)} + \ldots + c_1\dot{\hat{\eta}}_d + c_0\hat{\eta}_d = -\left(P_{k-1}r^{(k-1)} + \ldots + P_1\dot{r} + P_0 r\right) \tag{23}$$

where $c_{k-1}, c_{k-2}, \ldots, c_0$ is a set of Hurwitz polynomial coefficients, and $P_i$ is defined by

$$\begin{aligned}
&P_{k-1} = \left(I + c_{k-1}Q^{-1} + \ldots + c_0 Q^{-k}\right)\left(I + p_{k-1}Q^{-1} + \ldots + p_0 Q^{-k}\right)^{-1} - I \\
&P_{k-2} = c_{k-2}Q^{-1} + \ldots + c_0 Q^{-(k-1)} - (P_{k-1} + I)\left(p_{k-2}Q^{-1} + \ldots + p_0 Q^{-(k-1)}\right) \\
&\ldots \\
&P_1 = c_1 Q^{-1} + c_0 Q^{-2} - (P_{k-1} + I)\left(p_1 Q^{-1} + p_0 Q^{-2}\right) \\
&P_0 = c_0 Q^{-1} - (P_{k-1} + I)p_0 Q^{-1}
\end{aligned} \tag{24}$$

*Remark* Since the inversion of $Q$ is used, $Q$ must be nonsingular, which indicates that this method can be only used to zero dynamics without zero eigenvalues.

To evaluate the estimate precision, we define

$$e = \dot{\hat{\eta}}_d - Q\hat{\eta}_d - r \tag{25}$$

as the estimate error.

**Theorem 2** *The estimate error e satisfies* $e^{(k)} + c_{k-1}e^{(k-1)} + \ldots + c_0 e = 0$. *And since* $c_{k-1}, c_{k-2}, \ldots, c_0$ *is a set of Hurwitz polynomial coefficients, e will converge to zero asymptotically.*

*Proof* From (22) we have $y_d^{(k)} + p_{k-1} y_d^{(k-1)} + \ldots + p_1 \dot{y}_d + p_0 y_d = 0$. And from (20) we know that $r$ is a combination of $y_d$ and its derivatives, thus $r$ also satisfies $r^{(k)} + p_{k-1} r^{(k-1)} + \ldots + p_1 \dot{r} + p_0 r = 0$. According to (23) we have

$$\begin{aligned} &\hat{\eta}_d^{(k)} + c_{k-1} \hat{\eta}_d^{(k-1)} + \ldots + c_1 \dot{\hat{\eta}}_d + c_0 \hat{\eta}_d = -\left(P_{k-1} r^{(k-1)} + \ldots + P_1 \dot{r} + P_0 r\right) \\ &\Rightarrow \hat{\eta}_d^{(k+1)} + c_{k-1} \hat{\eta}_d^{(k)} + \ldots + c_1 \hat{\eta}_d^{(2)} + c_0 \dot{\hat{\eta}}_d = -\left(P_{k-1} r^{(k)} + \ldots + P_1 r^{(2)} + P_0 \dot{r}\right) \end{aligned} \tag{26}$$

Combining (24), (25), and (26), it follows that

$$\begin{aligned} e^{(k)} + c_{k-1} e^{(k-1)} + \ldots + c_0 e &= \hat{\eta}_d^{(k+1)} + c_{k-1} \hat{\eta}_d^{(k)} + \ldots + c_0 \hat{\eta}_d^{(1)} - Q\left(\hat{\eta}_d^{(k)} + c_{k-1} \hat{\eta}_d^{(k-1)} + \ldots + c_0 \hat{\eta}_d\right) - \left(r^{(k)} + c_{k-1} r^{(k-1)} + \ldots + c_0 r\right) \\ &= -\left(P_{k-1} r^{(k)} + \ldots + P_1 r^{(2)} + P_0 r^{(1)}\right) + Q\left(P_{k-1} r^{(k-1)} + \ldots + P_1 r^{(1)} + P_0 r\right) - \left(r^{(k)} + c_{k-1} r^{(k-1)} + \ldots + c_0 r\right) \\ &= (-I - P_{k-1}) r^{(k)} + (QP_{k-1} - c_{k-1} - P_{k-2}) r^{(k-1)} + \ldots + (QP_1 - c_1 - P_0) r^{(1)} + (QP_0 - c_0) r \\ &= -\left(I + c_{k-1} Q^{-1} + \ldots + c_0 Q^{-k}\right)\left(I + p_{k-1} Q^{-1} + \ldots + p_0 Q^{-k}\right)^{-1}\left(r^{(k)} + p_{k-1} r^{(k-1)} + \ldots + p_0 r\right) = 0 \end{aligned} \tag{27}$$

This completes the proof.

According to (18), the estimate of the equivalent input is defined by

$$\hat{u}_d = \alpha^{-1}\left(y_d^{(r)} - E\xi_d - F\hat{\eta}_d\right) \tag{28}$$

Denote the states tracking error $\tilde{z} = [\xi, \tilde{\eta}]^T$ with $\xi = \xi - \xi_d$, $\tilde{\eta} = \eta - \hat{\eta}_d$, and the output tracking error $\tilde{y} = y - y_d$.

**Theorem 3** *The control law*

$$u = -K\tilde{z} + \hat{u}_d \tag{29}$$

*with $A_1 - B_1 K$ be Hurwitz can realize asymptotic tracking for system (17).*

*Proof* Combining (20), (25), and (28), we will derive that

$$e = \dot{\hat{\eta}}_d - Q\hat{\eta}_d - r = \dot{\hat{\eta}}_d - R\xi_d - S\hat{\eta}_d - T\hat{u}_d \tag{30}$$

Define a new control input $v = u - u_d$, then from (16) we have

$$\begin{aligned} \tilde{y}^{(r)} &= E\xi + F\eta + \alpha u - y_d^{(r)} = E(\xi + \xi_d) + F(\tilde{\eta} + \hat{\eta}_d) + \alpha(v + \hat{u}_d) - y_d^{(r)} = E\xi + F\tilde{\eta} + \alpha v \\ \dot{\tilde{\eta}} &= R\xi + S\eta + Tu - \dot{\hat{\eta}}_d = R(\xi + \xi_d) + S(\tilde{\eta} + \hat{\eta}_d) + T(v + \hat{u}_d) - \dot{\hat{\eta}}_d \\ &= R\xi + S\tilde{\eta} + Tv - (\dot{\hat{\eta}}_d - R\xi_d - S\hat{\eta}_d - T\hat{u}_d) = R\xi + S\tilde{\eta} + Tv - e \end{aligned} \tag{31}$$

which can be written as $\dot{\tilde{z}} = A_1 \tilde{z} + B_1 v - [0_r; e]$ according to (17). Substituting $v = u - \hat{u}_d = -K\tilde{z}$ into it yields $\dot{\tilde{z}} = (A_1 - B_1 K)\tilde{z} - [0_r; e]$. Combine that $A_1 - B_1 K$ is

Hurwitz and $\lim_{t\to\infty} e \to 0$ from Theorem 2 we can obtain that $\lim_{t\to\infty} \tilde{z} \to 0$. This completes the proof.

Consider the linearized model (3), the equivalent input and equivalent states are defined by

$$\begin{aligned} \dot{s}_d &= v_d\,,\ \ \dot{v}_d = 0.7171\theta_d + 0.09756u_d \\ \theta_d &= \omega_d,\ \ \dot{\omega}_d = 7.888\theta_d + 0.07317u_d \end{aligned} \tag{32}$$

In case 1, $s$ and $v$ are the external states. From the first two equations of (32) we have $u_d = (\ddot{s}_d - 0.7171\theta_d)/0.09756$. Substituting it into the last two equations of (32) yields

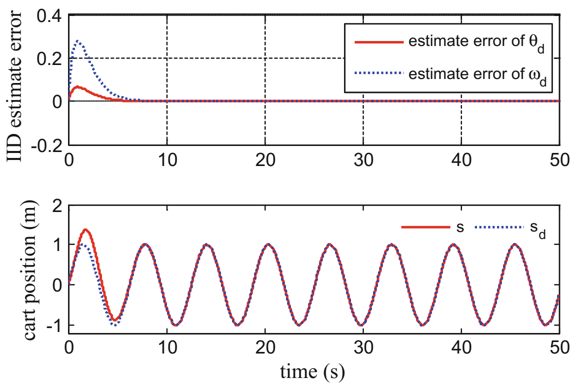
$$\begin{bmatrix} \theta_d \\ \dot{\omega}_d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 7.3502 & 0 \end{bmatrix} \begin{bmatrix} \theta_d \\ \omega_d \end{bmatrix} + \begin{bmatrix} 0 \\ 0.75\ddot{s}_d \end{bmatrix} \tag{33}$$

This is the internal dynamics with respect to output $s$. Since matrix $[0\ ,\ 1\ ;\ 7.3502\ ,\ 0]$ has eigenvalues $\pm 2.7111$, the internal dynamics is unstable. We use (23) to estimate the IID, where $p_0 = 1, p_1 = 0, r(t) = [0;\ -0.75\sin t]$. Choose $c_0 = 1, c_1 = 2$, using (24) $P_i$ is calculated as $P_0 = [-0.2395, 0; 0, -0.2395]$, $P_1 = [0, 0.2395; 1.7605, 0]$. So the IID estimator is designed as

$$\begin{bmatrix} \ddot{\theta}_d + 2\dot{\theta}_d + \theta_d \\ \ddot{\hat{\omega}}_d + 2\dot{\hat{\omega}}_d + \hat{\omega}_d \end{bmatrix} = \begin{bmatrix} 0.1796\ \cos\ t \\ -0.1796\ \sin\ t \end{bmatrix} \tag{34}$$

This together with (29) consists of the control law. Choose the same control gain as before and the initial value $\theta_d(0) = \dot{\theta}_d(0) = \hat{\omega}_d(0) = \dot{\hat{\omega}}_d(0)$. The simulation result is shown in Fig. 6, from which we can see that the IID estimator error converges to zero as well as the cart position tracking error within 10 s.

**Fig. 6** ID estimator error and cart position via stable system center



In case 2, $\theta, \omega$ are the external states. From the last two equations of (32) we have $u_d = (\dot{\theta}_d - 7.888\theta_d)/0.07317$. Substituting it into the first two equations of (32) yields

$$\begin{bmatrix} \dot{s}_d \\ \dot{v}_d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_d \\ v_d \end{bmatrix} + \begin{bmatrix} 0 \\ -9.8\theta_d + 1.333\dot{\theta}_d \end{bmatrix} \tag{35}$$

This is the internal dynamics with respect to output $\theta$. Since matrix $[0, 1; 0, 0]$ has zero eigenvalue, it cannot be inverted. Thus stable system center method cannot be used in this case.

## 6 Conclusion

The approximate tracking method is simplest and is suitable for slow varying trajectory tracking, but tracking error increases as the tracking frequency rises. Output regulation and stable system center method can achieve asymptotic tracking. They both need to know the exosystem of the reference output, and both focus on transforming the tracking problem into stabilizing problem. Output regulation method constructs regulator equation and calculates the equivalent input and equivalent states. Stable system center method is based on the system normal form and tries to estimate the ideal internal dynamics, but it can be only used to system with zero dynamics that has no zero eigenvalues. Although simulation results exhibit good performance, we should notice that these three methods are all based on the linearized model. Exact tracking for wider range and higher frequency reference signals still remain a challenge to the nonlinear inverted pendulum.

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